

## Waves in turbulent stably stratified shear flow

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Two approaches for the identification of internal gravity waves in sheared and un-sheared homogeneous stratified turbulence are investigated. First, the phase angle between the vertical velocity and density fluctuations is considered. It was found, however, that a continuous distribution of the phase angle is present in weakly and strongly stratified flow. Second, a projection onto the solution of the linearized inviscid equations of motion of un-sheared stratified flow is investigated. It was found that a solution of the fully nonlinear viscous Navier-Stokes equations can be represented by the linearized inviscid solution. The projection yields a decomposition into vertical wave modes and horizontal vortical modes.

### 1. Introduction

An important problem in geophysical fluid mechanics is the characterization of turbulence and wave motion in stably stratified flows. Fluid motion can occur as a result of either of these phenomena and being able to separate the motions associated with each should lead to better understanding and predictability of the flow. Stewart (1969) listed criteria that might be used to distinguish between internal wave motion and turbulence. The first distinction noted was that wave motion satisfies linear equations, whereas turbulence is inherently nonlinear. However, when both waves and turbulence are present, the motions are coupled nonlinearly and it is unclear how to extract the wave component of the flow. Secondly, the process by which energy is transported is different. In turbulence, energy is advected at the speed of the motion, whereas waves transport energy through pressure-velocity correlations, usually at a group velocity that is faster than the particle velocity. However, pressure-velocity correlations also exist in turbulence when waves are not important. Lastly, Stewart noted the difference between turbulence and waves with regard to mixing. Except when they break, waves do not produce mixing. Although they can transport momentum, they cannot transport scalars. Thus the scalar flux  $\overline{u_2 \rho}$ , where  $u_2$  is the vertical velocity component, should be large in regions dominated by turbulence and small where waves predominate. Furthermore, the relative phase of vertical velocity fluctuations  $u_2$  and density fluctuations  $\rho$  is different for waves and turbulence. For stably stratified flows, in-phase motion between  $u_2$  and  $\rho$  corresponds to down-the-gradient turbulent transport, while  $180^\circ$  out-of-phase motion is associated with counter-gradient turbulent transport. For wave motions,  $u_2$  and  $\rho$  have a phase difference of  $90^\circ$  and there is no mean correlation between them.

Stewart (1969) concluded that this last distinction held the greatest promise for separating waves and turbulence and this criterion has been used extensively since. For example, Stillinger, Helland, & Van Atta (1983) felt that their un-sheared stably stratified decaying turbulence "had been completely converted to random internal wave motions" when  $\overline{u_2 \rho}$  became zero. However, Lienhard & Van Atta (1990) pointed out that  $\overline{u_2 \rho}$  can

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be zero as a result of co-gradient and counter-gradient fluxes at different scales of motion cancelling each other out. More careful diagnosis requires examination of the cospectrum of  $\overline{u_2 \rho}$  as a function of wavenumber, as originally proposed by Stewart (1969). Defining the cospectrum  $Co$  and quadrature spectrum  $Qu$  as

$$Co_{u_2 \rho}(k_1, x_2) = Re(\Sigma_{k_3} \tilde{u}_2^*(k_1, x_2, k_3) \tilde{\rho}(k_1, x_2, k_3)) \quad (1.1)$$

$$Qu_{u_2 \rho}(k_1, x_2) = Im(\Sigma_{k_3} \tilde{u}_2^*(k_1, x_2, k_3) \tilde{\rho}(k_1, x_2, k_3)) , \quad (1.2)$$

where the tildes indicate Fourier transformed quantities, the phase angle  $\phi_{u_2 \rho}$  between vertical velocity  $u_2$  and density  $\rho$  is given by

$$\phi_{u_2 \rho} = atan\left(\frac{Qu_{u_2 \rho}}{Co_{u_2 \rho}}\right). \quad (1.3)$$

The above spectral quantities (or similar measures in terms of other wavenumber components) have been used in evaluating both experimental and computational data on stratified flows. McBean & Miyake (1972) used measurements in the atmospheric surface layer to tentatively conclude that wave motions may be important at low frequencies in stably stratified flow. Komori, Ueda, Ogino & Mizushima (1983) felt that a significant fraction of the motion in their stably stratified open-channel flow experiment was wave-like based on the phase angles measured. In contrast, data from experiments in both unsheared (Lienhard & Van Atta 1990) and sheared (Piccirillo & Van Atta 1997) stably stratified homogeneous turbulence indicate no evidence of wavelike motion based on examination of the phase angle. Analysis of direct numerical simulations of similar sheared homogeneous stratified turbulence (Holt, Koseff & Ferziger 1992) also indicates that even for strong stratification there is no band of wavenumbers with  $\phi_{u_2 \rho} \approx 90^\circ$ .

Riley, Metcalfe, and Weissman (1981) proposed a different method for separating wave-like and turbulent motions. They used the Craya (1958) decomposition to split the turbulent velocity field associated with each wavenumber into two solenoidal components, one normal to the wavenumber vector and the gravity vector, and the other orthogonal to this component and the wavenumber vector. For small amplitudes, this second component satisfies the linear propagation equation for internal gravity waves and is thus identified as the "wave" component of the motion. The other component consists of quasi-horizontal motions containing all the vertical vorticity and is identified as "turbulence". This decomposition only splits the flow into propagating and non-propagating parts in the limit of zero Froude number. For small but finite Froude number Staquet and Riley (1989) proposed a generalization of this decomposition using Ertel's (1942) Theorem for potential vorticity. However, this generalization is invalid when the density gradient is zero or unbounded and therefore cannot be used for turbulent flows.

Despite this shortcoming, Herring and Métais (1989) and Métais and Herring (1989) used the original Riley *et al.* decomposition to split their numerically simulated turbulent flow fields into "wave" and "turbulent" components. They acknowledge the deficiencies of this approximation, noting 1) that "a proper definition of waves should include the density field, and its phase relative to the 'wave'-component of the velocity" and 2) their non-zero Froude number. However, the "turbulent" components of their flows do not show oscillations that scale with the Brunt-Väisälä frequency; such oscillations are observed in the wave component of the flows. This suggests a weak interaction between the components and perhaps adequacy of the decomposition.

The prototypical example of homogeneous turbulent stratified shear flow with uniform stable vertical stratification  $S_\rho = \partial \rho / \partial x_2$  and uniform vertical shear  $S = \partial U / \partial x_2$  is the

simplest flow that contains both shear and stratification. It has been studied extensively in the past due to its geophysical significance. Experimental investigations include Rohr, Itsweire, Helland & Van Atta (1988) and Piccirillo & Van Atta (1997). Numerical simulations include the work by Gerz, Schumann & Elghobashi (1989), Holt, Koseff & Ferziger (1992), Jacobitz, Sarkar & Van Atta (1997), and Jacobitz (2000). Turbulence in decaying stratified turbulence without shear has been investigated by Métais & Herring (1989), Lienhard & Van Atta (1990), Yoon & Warhaft (1990), and Briggs, Ferziger, Koseff & Monismith (1998).

In this study, possible ways to decompose the fluid motion into turbulence and wave components are investigated in direct numerical simulations of both sheared and un-sheared homogeneous stratified turbulence. Both the phase angle between the vertical velocity and density and projections onto eigensolutions of the linearized governing equations are examined.

In the following section, the numerical simulations used in the present study are introduced. In sections 3 and 4, the phase angle results in sheared and un-sheared stably stratified turbulence are presented. In section 5, a turbulence-wave decomposition based on the linear inviscid equations of motion is applied to the numerical data. Results are summarized in section 6.

## 2. The numerical simulations

The current study is based on the results of five direct numerical simulations of sheared homogeneous stably stratified turbulence and two direct numerical simulations of un-sheared decaying homogeneous stably stratified turbulence.

In the direct numerical simulations, all dynamically important scales of the velocity, density and pressure fields are resolved and no turbulence models are introduced. A spatial discretization is first performed to obtain a semi-discrete system of ordinary differential equations from the original system of partial differential equations. An integration of the system of ordinary differential equations is then performed to advance the solution in time. The spatial discretization is accomplished by a spectral collocation method. The temporal advancement is accomplished by a fourth-order Runge-Kutta scheme. A computational grid overlaying a cube of length  $2\pi$  was used with  $256^3$  points. The initial conditions are taken from a separate simulation of isotropic turbulence without density fluctuations, which was allowed to develop for approximately one eddy turnover time. The energy spectrum of the initial field peaks at a wave number  $k = 13$  and the resulting vertical integral scale, computed as the vertical integral of the autocorrelation of the vertical velocity component, is  $L = 0.174$ , compared to the box size  $2\pi$ . The initial value of the Taylor micro-scale Reynolds number  $Re_\lambda = 45$  is fixed in all simulations.

Figure 1 shows the evolution of the normalized turbulent kinetic energy  $K/K_0$  for sheared stably stratified turbulence with Richardson numbers  $Ri = 0$ ,  $Ri = 0.1$ ,  $Ri = 0.2$ ,  $Ri = 0.5$ , and  $Ri = 1.0$ . Here the Richardson number is given by  $N^2/S^2$ , where  $N$  is the Brunt-Väisälä frequency, given by  $\sqrt{(-g/\rho_0)\partial\rho/\partial y}$ . Initially, the turbulent kinetic energy decays as a result of the absence of Reynolds shear stress  $\overline{u_1 u_2}$  in the isotropic initial condition. For simulations with small values of the Richardson number, the turbulent kinetic energy eventually grows with nondimensional time  $St$ . For simulations with large values of the Richardson number, however, the turbulent kinetic energy continues to decay, with the stratification overwhelming the turbulence production by the mean shear.

Figure 2 shows the evolution of the normalized turbulent kinetic energy  $K/K_0$  for

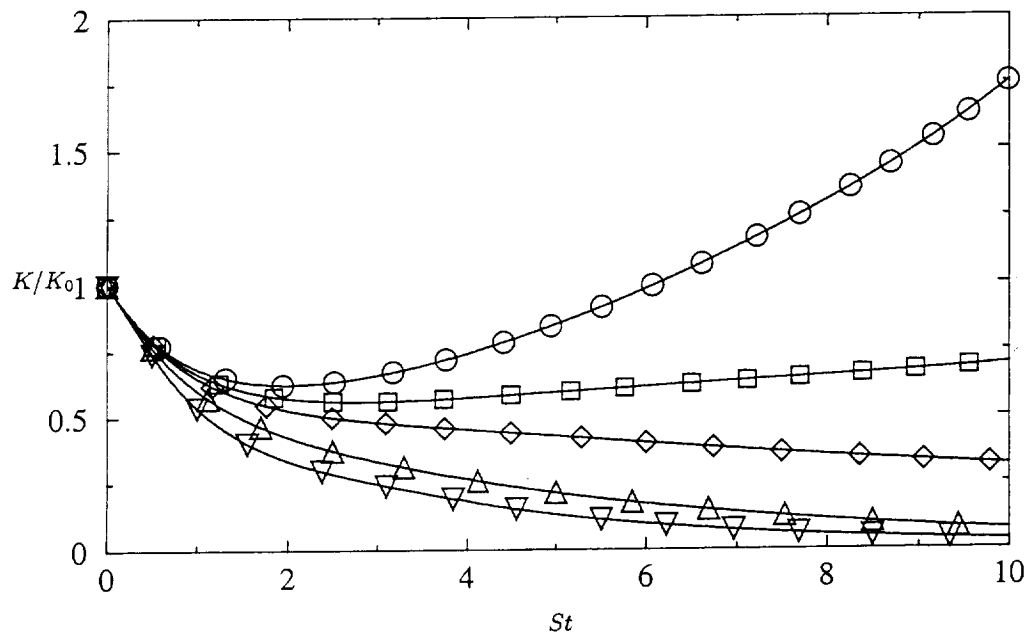


FIGURE 1. Evolution of the normalized turbulent kinetic energy  $K/K_0$  in sheared stratified turbulence with Richardson numbers 0 ( $\circ$ ), 0.1 ( $\square$ ), 0.2 ( $\diamond$ ), 0.5 ( $\Delta$ ), and 1.0 ( $\nabla$ ).

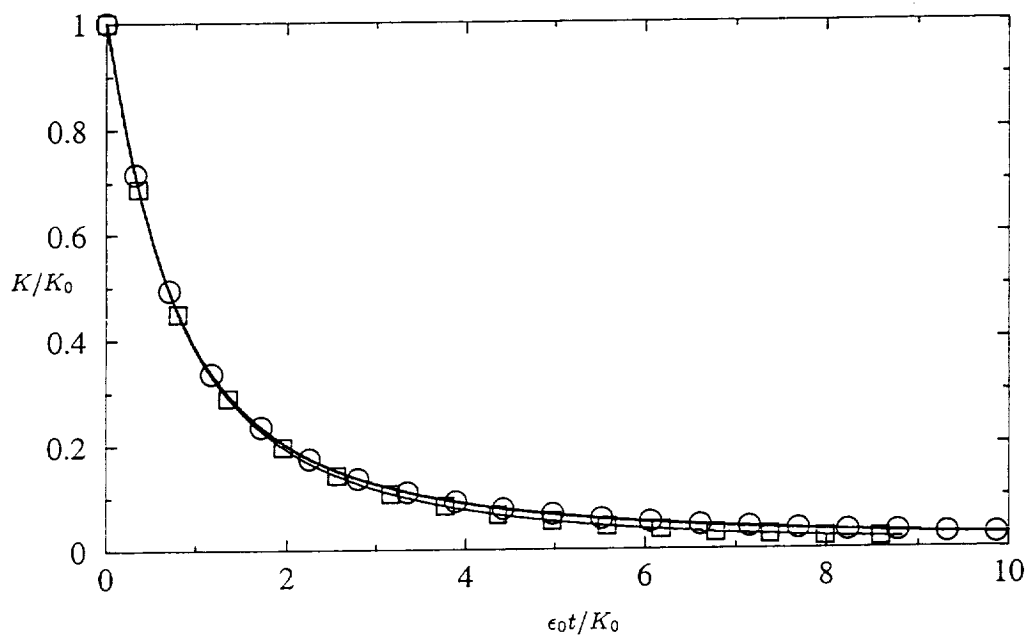


FIGURE 2. Evolution of the normalized turbulent kinetic energy  $K/K_0$  in unsheared stratified turbulence with initial Froude numbers 64 ( $\circ$ ) and 6.4 ( $\square$ ).

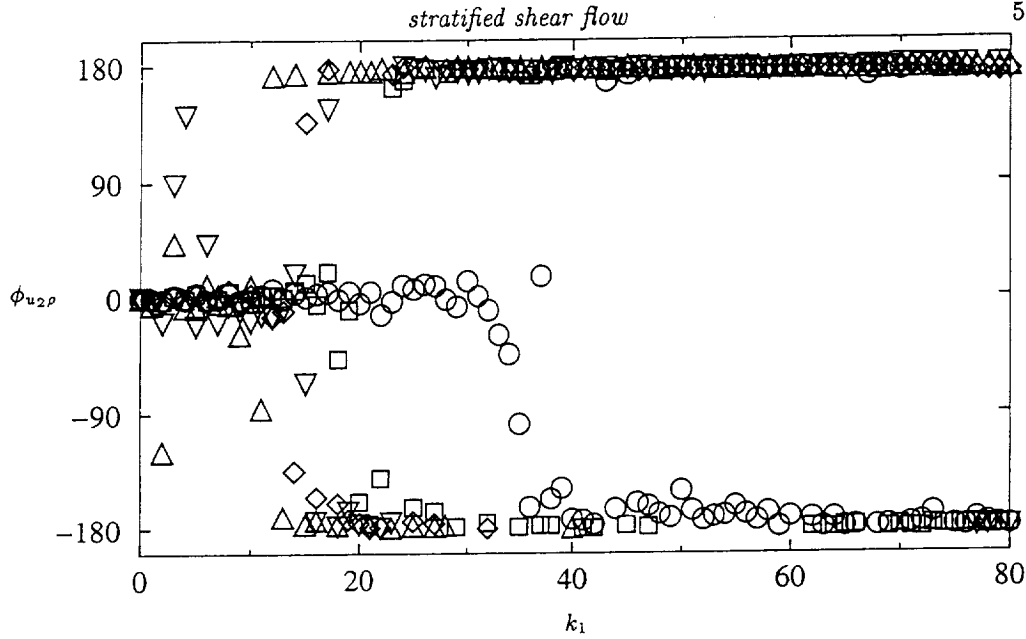


FIGURE 3. Spectrum of the phase angle  $\phi_{u_2\rho}$  in sheared stratified turbulence at  $St = 5$  with Richardson numbers 0 ( $\circ$ ), 0.1 ( $\square$ ), 0.2 ( $\diamond$ ), 0.5 ( $\triangle$ ), and 1.0 ( $\nabla$ ).

unsheared stably stratified turbulence with initial Froude numbers  $Fr = 64$  and  $Fr = 6.4$ , where  $Fr = \sqrt{u_2^2}/(LN)$ . A slightly stronger decay of the turbulent kinetic energy is found for the more strongly stratified simulation.

### 3. Phase angle in sheared stably stratified turbulence

In this section, the phase angle in sheared stably stratified turbulence is discussed. The Richardson number  $Ri$  is varied from  $Ri = 0$ , corresponding to unstratified shear flow, to  $Ri = 1$ , corresponding to strongly stratified shear flow.

Figure 3 shows the spectrum of the phase angle  $\phi_{u_2\rho}$  between the vertical velocity  $u_2$  and the density  $\rho$  at non-dimensional time  $St = 5$ . In the unstratified simulation with  $Ri = 0$  ( $\circ$  symbols), phase angles  $\phi_{u_2\rho} \approx 0$  are found for small wave numbers  $k_1$  and phase angles  $\phi_{u_2\rho} \approx \pm 180^\circ$  are found for large values of  $k_1$ . The transition from  $\phi_{u_2\rho} \approx 0$  to  $\phi_{u_2\rho} \approx \pm 180^\circ$  occurs at a wave number  $k_1 \approx 35$ . At this wave number, the cospectrum  $Co_{u_2\rho}$  crosses zero and changes sign. As the Richardson number is increased, the transition wave number decreases to about  $k_1 \approx 20$  for  $Ri = 0.1$ ,  $k_1 \approx 17$  for  $Ri = 0.2$ ,  $k_1 \approx 10$  for  $Ri = 0.5$ , and  $k_1 \approx 5$  for  $Ri = 1$ . Phase angles  $\phi_{u_2\rho} \approx \pm 90^\circ$ , indicating possible internal wave motion, are observed only in the strongly stratified simulations with  $Ri = 0.5$  and  $Ri = 1$  and only for a few scattered wavenumbers, not over a region of wavespace. Again, these isolated instances of  $90^\circ$  phase angles are associated with zero-crossings of the associated cospectrum, rather than a region in wavespace exhibiting wavelike behavior.

Figure 4 shows the probability distribution of the phase angle  $\phi_{u_2\rho}$  over an instantaneous flow field. The phase angle distribution of the unstratified simulation with  $Ri = 0$  has a slight maximum at  $\phi_{u_2\rho} = 0$ , indicating a very modest predominance of down-

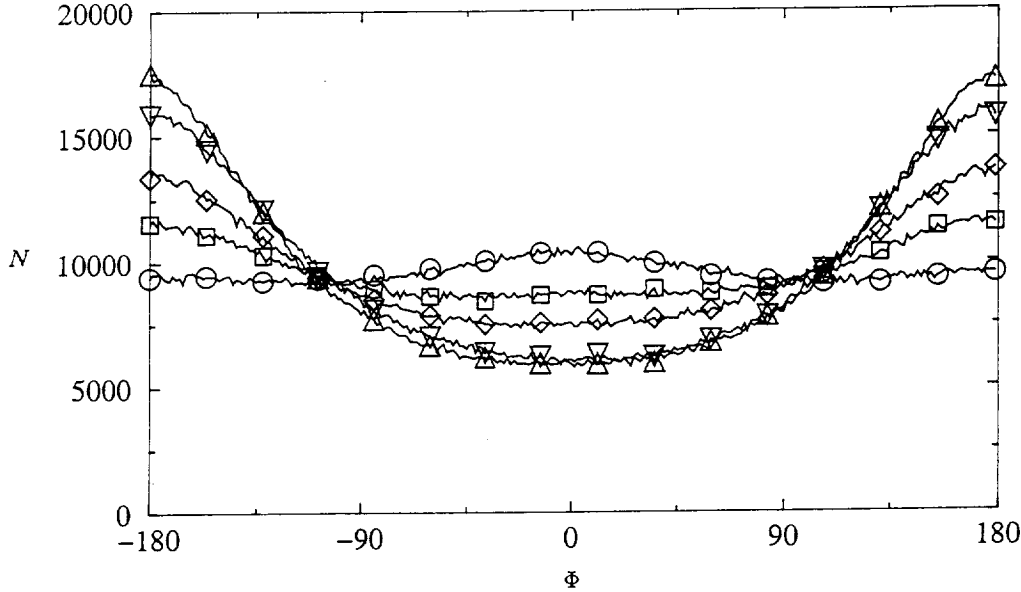


FIGURE 4. Distribution of the phase angle  $\phi_{u_2\rho}$  in sheared stratified turbulence at  $St = 5$  with Richardson numbers 0 ( $\circ$ ), 0.1 ( $\square$ ), 0.2 ( $\diamond$ ), 0.5 ( $\triangle$ ), and 1.0 ( $\nabla$ ).

gradient mixing. As the Richardson number is increased, the maximum of the phase angle distribution is found at  $\phi_{u_2\rho} = \pm 180^\circ$ , corresponding to counter-gradient mixing. The largest contribution to  $\phi_{u_2\rho} = \pm 180^\circ$  is found in the  $Ri = 0.5$  case, which also shows the strongest counter-gradient mixing coefficient. For all cases, a continuous phase angle distribution is observed. There are no local peaks apparent around  $\phi_{u_2\rho} = \pm 90^\circ$  that would suggest regions of wavelike behavior distinct from the background turbulence.

In order to obtain a more complete picture of phase angle distributions in turbulent stratified flow, figure 5 shows the distribution of the phase angle  $\phi_{u_1u_2}$  between downstream  $u_1$  and vertical  $u_2$  velocity components, again at  $St = 5$ . The distribution is relatively unaffected by the Richardson number variation. It shows strong peaks around  $\phi_{u_1u_2} = 0$  and around  $\phi_{u_1u_2} = \pm 180^\circ$ . For modes with  $k_3 = 0$ , the continuity equation in wave space requires that the Fourier coefficients  $\tilde{u}_1$  and  $\tilde{u}_2$  are in the same direction, corresponding to  $\phi_{u_1u_2} = 0$ , or in opposite directions, corresponding to  $\phi_{u_1u_2} = \pm 180^\circ$ . The peaks therefore are a result of two-dimensional modes.

#### 4. Phase angle in unsheared stably stratified turbulence

In this section, the phase angle in unsheared decaying stably stratified turbulence is discussed. A weakly stratified case with initial Froude number  $Fr = 64$  is compared to a more strongly stratified case with  $Fr = 6.4$ .

The spectrum of the phase angle  $\phi_{u_2\rho}$  after about 10 eddy-turnover times is shown in figure 6. The weakly stratified case with  $Fr = 64$  has  $\phi_{u_2\rho} \approx 0$  for all  $k_1$ , corresponding to down-gradient flux. The case with  $Fr = 6.4$ , however, shows  $\phi_{u_2\rho} \approx \pm 180^\circ$  for wave numbers larger than about  $k_1 = 30$ , corresponding to counter-gradient mixing. As with the sheared cases, there is no band in wavespace with wavelike behavior.

The phase angle distribution over the instantaneous field at the same time is shown in

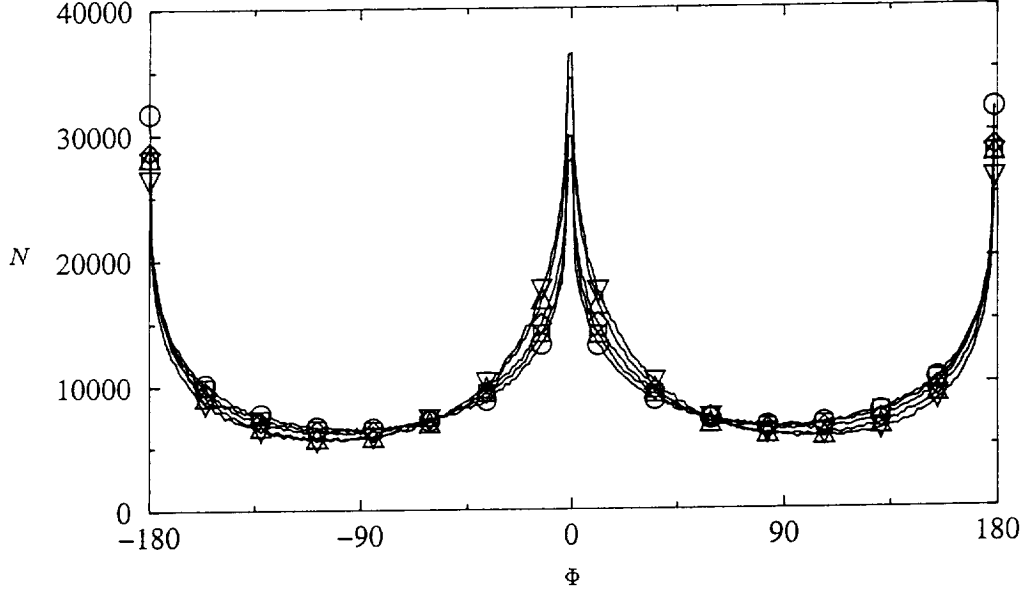


FIGURE 5. Distribution of the phase angle  $\phi_{u_1, u_2}$  in sheared stratified turbulence at  $St = 5$  with Richardson numbers 0 ( $\circ$ ), 0.1 ( $\square$ ), 0.2 ( $\diamond$ ), 0.5 ( $\triangle$ ), and 1.0 ( $\nabla$ ).

figure 7. The distribution of the weakly stratified case shows a clear maximum around  $\phi_{u_2\rho} = 0$ . However, the phase angles are widely distributed, which perhaps would not have been anticipated given the distribution in figure 6, which shows averaged phase angles at a given wave number. The distribution of the  $Fr = 6.4$  case shows a maximum around  $\phi_{u_2\rho} = \pm 180^\circ$ , corresponding to counter-gradient mixing. The distribution of phase angles between downstream velocity  $u_1$  and vertical velocity  $u_2$  is very similar to that of the sheared case, with strong peaks at  $0^\circ$  and  $\pm 180^\circ$  and a weak dependence on the strength of the stratification.

### 5. Normal mode analysis

A normal mode analysis of the linearized inviscid equations of motion for the un-sheared flow is performed. The direct numerical simulation data is then projected onto the eigensolution in order to extract a possible linear wave motion present in the data.

The analysis is based on the following linearized inviscid equations of motion:

$$\frac{\partial \rho}{\partial t} = -S_\rho u_2$$

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x_i} - \frac{g}{\rho_0} \rho \delta_{i2}$$

The pressure is eliminated from the equations using the continuity equation. The equations are transformed into Fourier space and take the following form:

$$\frac{\partial \bar{\rho}}{\partial t} = -\frac{g}{\rho_0} S_\rho \bar{u}_2 = N^2 \bar{u}_2$$

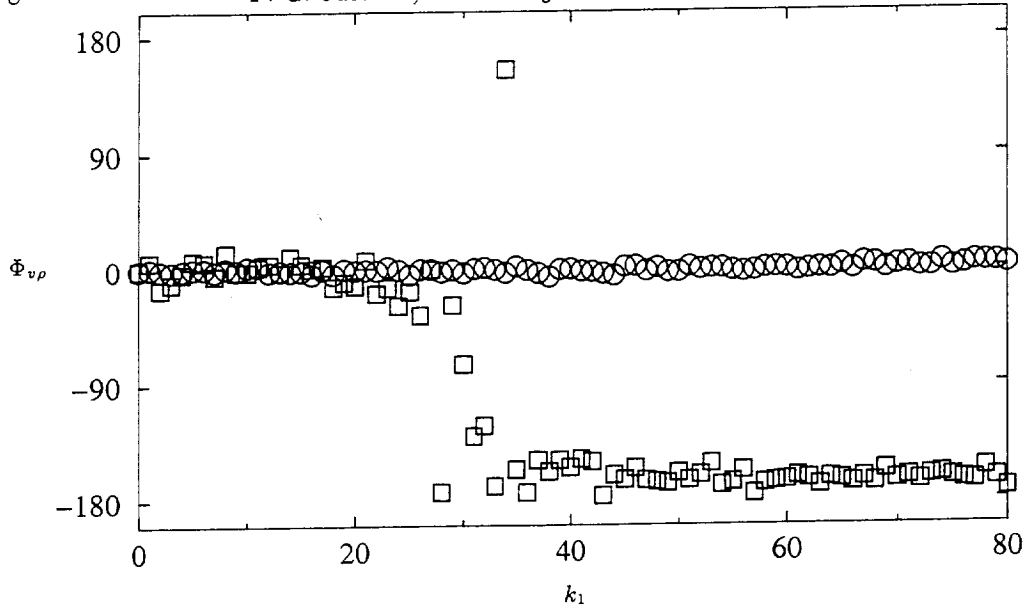


FIGURE 6. Spectrum of the phase angle  $\phi_{u_2\rho}$  in unsheared stratified turbulence after about 10 eddy-turnover times with initial Froude numbers 64 ( $\circ$ ) and 6.4 ( $\square$ ).

$$\frac{\partial \tilde{u}_i}{\partial t} = \bar{\rho} \left( \frac{k_i k_2}{k^2} - \delta_{i2} \right).$$

Normal modes of the form

$$\begin{pmatrix} \tilde{\rho} \\ \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{pmatrix} = \begin{pmatrix} \hat{\rho} \\ \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} \exp i(k_j x_j + \omega t)$$

are introduced and lead to the following system of equations:

$$\begin{pmatrix} i\omega & 0 & S_\rho & 0 \\ -\frac{g}{\rho_0} \frac{k_1 k_2}{k^2} & i\omega & 0 & 0 \\ -\frac{g}{\rho_0} \left( \frac{k_2^2}{k^2} - 1 \right) & 0 & i\omega & 0 \\ -\frac{g}{\rho_0} \frac{k_2 k_3}{k^2} & 0 & 0 & i\omega \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{pmatrix} = 0$$

From this system of equations, the following dispersion relation is obtained:

$$\omega^2 = 0 \quad \omega = \pm \sqrt{D}$$

Here,  $D$  takes the following value:

$$D = N^2 \frac{k_1^2 + k_3^2}{k^2}$$



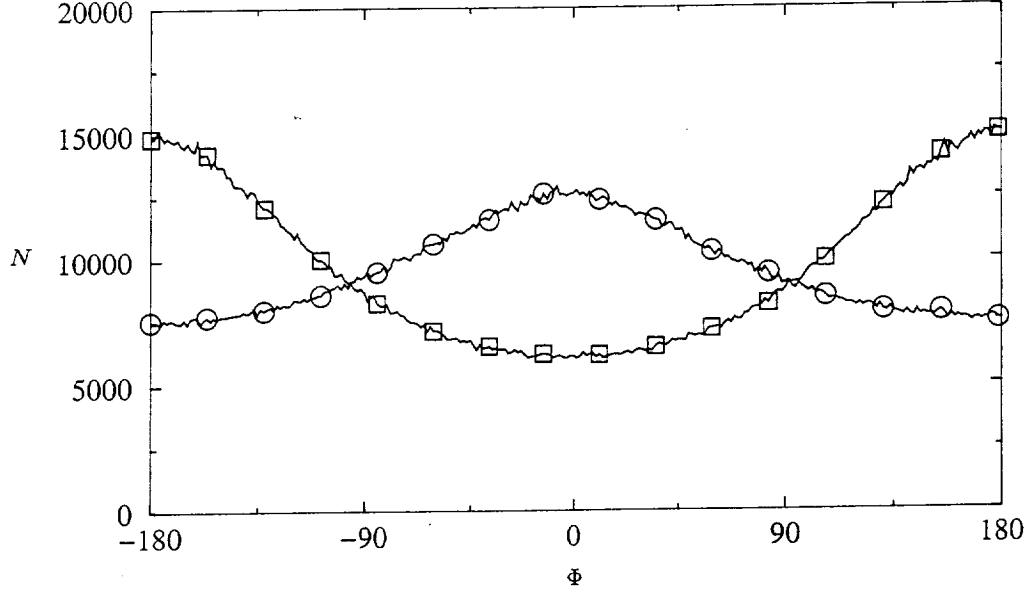


FIGURE 7. Distribution of the phase angle  $\phi_{u_2\rho}$  in unsheared stratified turbulence after about 10 eddy-turnover times with initial Froude numbers 64 (○) and 6.4 (□).

The following eigenvectors are obtained:

$$\mathbf{e}_{1,3} = \begin{pmatrix} S_\rho(k_1^2 + k_3^2) \\ \mp i k_1 k_2 \sqrt{D} \\ \pm i(k_1^2 + k_3^2) \sqrt{D} \\ \mp i k_2 k_3 \sqrt{D} \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The solution from direct numerical simulations can now be expressed in terms of the eigenvectors:

$$\mathbf{a}_{DNS} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 + a_4 \mathbf{e}_4$$

Here, the components  $a_2$  and  $a_4$  describe horizontal vortical motion. The components  $a_1$  and  $a_3$  define an upper bound for the wave motion present in the field. Note that any DNS data, except that for  $k_1$  and  $k_3$  both zero, can be represented by a choice of complex  $a_1$ ,  $a_2$ ,  $a_3$ , and  $a_4$ . The coefficients are found by multiplication with the complex conjugate of the eigenvectors of the adjoint problem.

Note that the solution to the linearized governing equations is also used in Rapid Distortion Theory. The analytical solution to the equations presented at the beginning of this section was developed by Hanazaki and Hunt (1996). The solutions of these linearized equations show an impressive degree of similarity to solutions of the full nonlinear problem and capture many of the distinctive behaviors of stably stratified turbulence. Remarkably, Hanazaki and Hunt (2002) have extended this analysis to include the case of uniformly sheared stratified turbulence as well. Presumably this solution could be used to provide guidance on how to decompose the sheared flow fields, but the difficulties encountered above would still be present (namely, all the turbulent motion could be represented by the eigenvectors of the linearized system and even after eliminating

horizontal motions containing the vertical vorticity the remaining “wave” motion could still contain a turbulent component).

## 6. Summary

In this study, the phase angle  $\phi_{u_2\rho}$  between vertical velocity  $u_2$  and density  $\rho$  was computed from direct numerical simulations of sheared and unsheared homogeneous stratified turbulence. A broad distribution of the phase angle was found that is consistent with observed down-gradient mixing for weakly stratified flow and counter-gradient mixing for strongly stratified flow. However, the broad distribution hides any internal wave signature that may be present in the flow.

A decomposition based on linear analysis has been proposed for unsheared decaying stratified turbulence. The flow fields are decomposed into horizontal vortical motions and vertical wave motions. However, there may still be some turbulent motion contained in the wave field. In agreement with Stewart (1969) we find that “there is probably no really clear-cut distinction between turbulence and waves”.

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